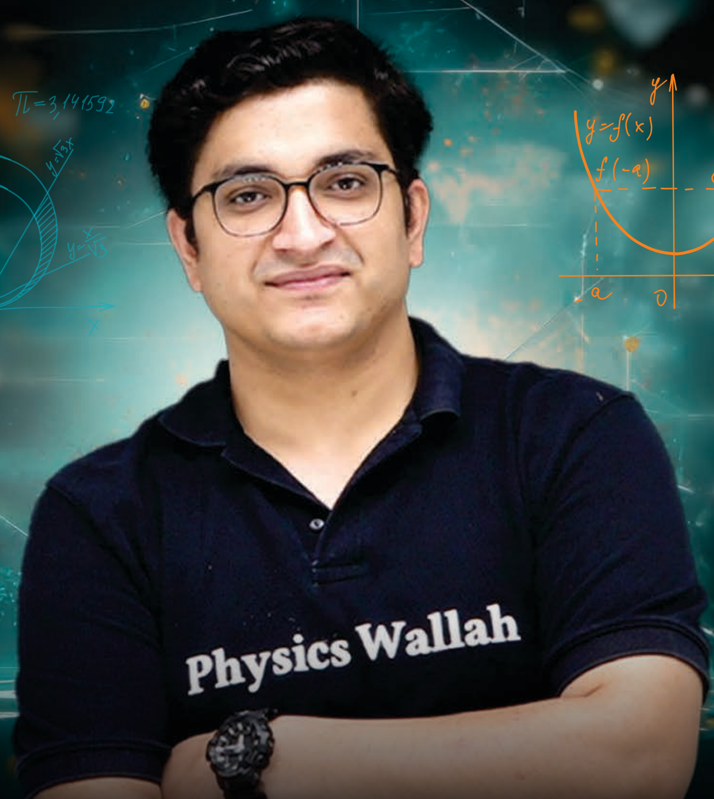
“Hello Hello Nazrein Calculus Par”

CALCULUS CORE

Fear No More

Basic to Advanced Level
Detailed Theory with Topic wise Solved Examples
JEE Main & IIT-JEE Advanced Exercises
Sachin Sir Special (S^3)
Critical Thinking Questions (CTQ)



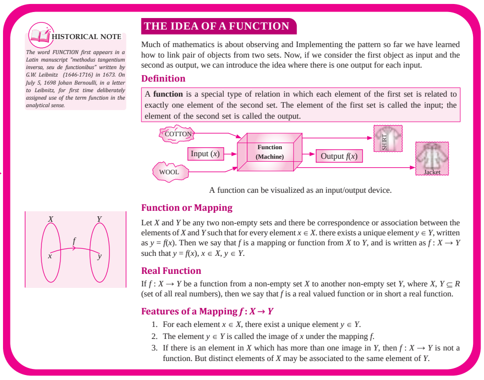
Sachin Jakhar

Contents

HOW TO USE THIS BOOK IN BEST WAY

DETAILED THEORY

All topics are arranged in best possible way to enhance your learning of the chapter. Go through each point/heading/ subheading of the theory.


HISTORICAL NOTE

The word FUNCTION first appears in a Latin manuscript "methodus tangentium inversa, seu de functionibus" written by G.W. Leibnitz (1646-1716) in 1673. On July 5, 1698 Johan Bernoulli, in a letter to Leibnitz, for first time deliberately assigned use of the term function in the analytical sense.

THE IDEA OF A FUNCTION

Much of mathematics is about observing and Implementing the pattern so far we have learned how to link pair of objects from two sets. Now, if we consider the first object as input and the second as output, we can introduce the idea where there is one output for each input.

Definition

A **function** is a special type of relation in which each element of the first set is related to exactly one element of the second set. The element of the first set is called the input; the element of the second set is called the output.

A function can be visualized as an input/output device.

Function or Mapping

Let X and Y be any two non-empty sets and there be correspondence or association between the elements of X and Y such that for every element $x \in X$. there exists a unique element $y \in Y$, written as $y = f(x)$. Then we say that f is a mapping or function from X to Y, and is written as $f : X \to Y$ such that $y = f(x)$, $x \in X$, $y \in Y$.

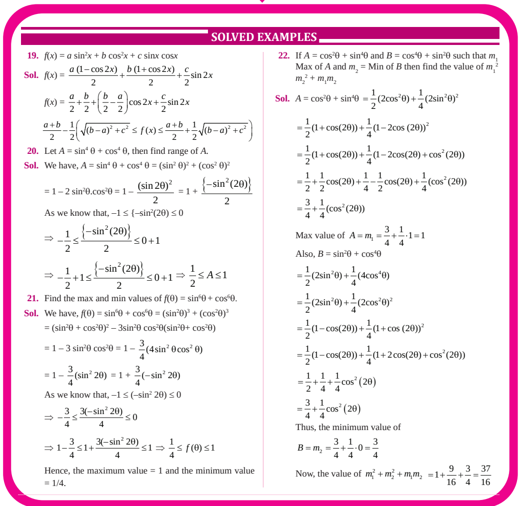
Real Function

If $f : X \to Y$ be a function from a non-empty set X to another non-empty set Y, where $X, Y \subseteq R$ (set of all real numbers), then we say that f is a real valued function or in short a real function.

Features of a Mapping $f : X \to Y$

1. For each element $x \in X$, there exist a unique element $y \in Y$.
2. The element $y \in Y$ is called the image of x under the mapping f.
3. If there is an element in X which has more than one image in Y, then $f : X \to Y$ is not a function. But distinct elements of X may be associated to the same element of Y.

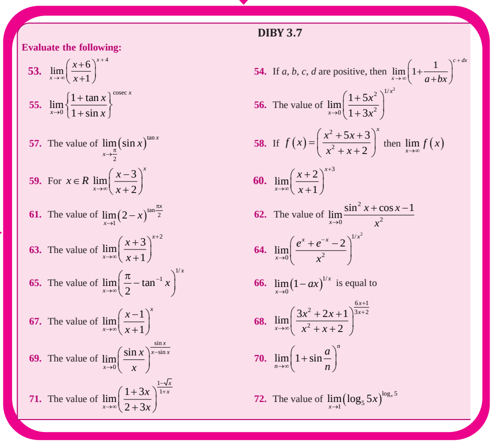
SOLVED EXAMPLES


SOLVED EXAMPLES

Solved Examples will help you in developing thought of question solving and application of concept covered. Observe the method & technique of question solving and keep that thing in mind as you proceed further.

DO IT BY YOURSELF (DIBY)

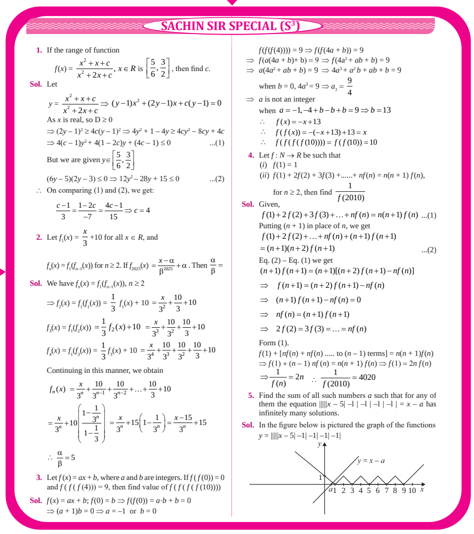
These are simple questions based on above concept covered. These DIBY (Do It By Yourself) Questions will help you to get strong hold on that particular topic. In case of any difficulty refer to detailed solutions given at end of the book.


DIBY 3.7

Evaluate the following:

53. $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}$
54. If a, b, c, d are positive, then $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}$
55. $\lim_{x\to 0}\left\{\frac{1+\tan x}{1+\sin x}\right\}^{\operatorname{cosec} x}$
56. The value of $\lim_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}$
57. The value of $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}$
58. If $f(x)=\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$ then $\lim_{x\to\infty} f(x)$
59. For $x \in R$ $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x$
60. $\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x+3}$
61. The value of $\lim_{x\to 1}(2-x)^{\tan\frac{\pi x}{2}}$
62. The value of $\lim_{x\to 0}\frac{\sin^2 x+\cos x-1}{x^2}$
63. The value of $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+2}$
64. $\lim_{x\to 0}\left(\frac{e^x+e^{-x}-2}{x^2}\right)^{1/x^2}$
65. The value of $\lim_{x\to\infty}\left(\frac{\pi}{2}-\tan^{-1}x\right)^{1/x}$
66. $\lim_{x\to 0}(1-ax)^{1/x}$ is equal to
67. The value of $\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^x$
68. $\lim_{x\to\infty}\left(\frac{3x^2+2x+1}{x^2+x+2}\right)^{\frac{6x+1}{3x+2}}$
69. The value of $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$
70. $\lim_{n\to\infty}\left(1+\sin\frac{a}{n}\right)^n$
71. The value of $\lim_{x\to\infty}\left(\frac{1+3x}{2+3x}\right)^{\frac{1-\sqrt{x}}{1+x}}$
72. The value of $\lim_{x\to 1}(\log_3 5x)^{\log_x 5}$

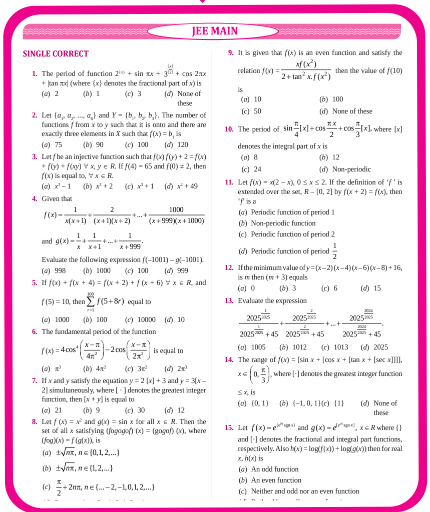
SACHIN SIR SPECIAL (S³)


SACHIN SIR SPECIAL (S³)

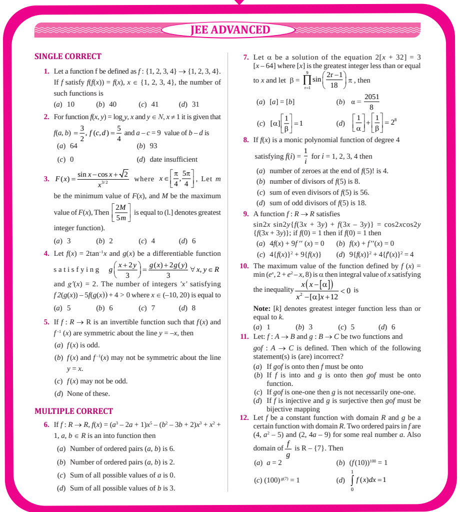
Here you will find amazing solved problems designed & curated by me. These problems will help you in increasing the canvas size of your mathematical aptitude and will boost your confidence to try new & challenging problems for JEE Advanced.

EXERCISES

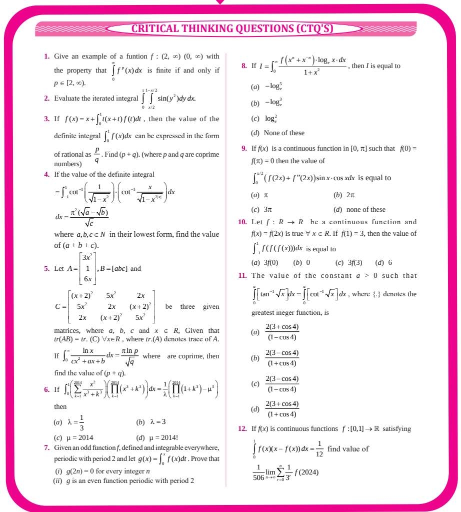
JEE MAIN



JEE ADVANCED



CRITICAL THINKING QUESTION (CTQ'S)



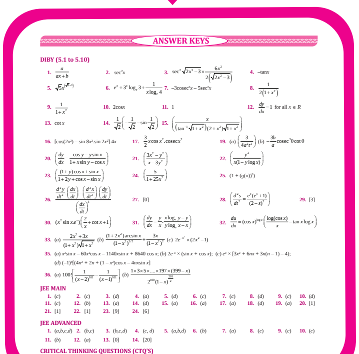
Here you will find questions based on latest pattern of JEE-Main's i.e. (Single correct & Numerical Type). I suggest to solve these questions, immediately after completion of theory and solved examples & DIBY.

Inclusion of every type of questions asked in IIT-JEE Advanced. I suggest to attempt these questions category-wise in different sittings. This exercise will surely fulfil the need of sufficiency for JEE Advanced.

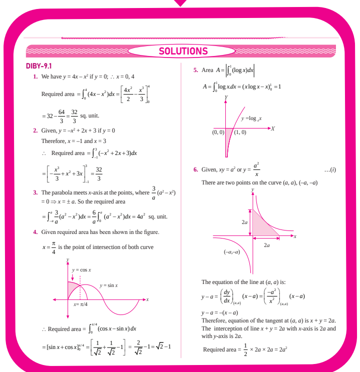
These exist a variety of questions in critical thinking to solve. If you have craving of problem solving & exploring other areas of mathematics (i.e. Olympiads) then this exercise is for you.

SOLUTIONS

ANSWER KEY



DETAILED SOLUTION OF EVERY QUESTION



Complete Answer key is provided at the end of each chapter.

Detailed solution of every question is provided at end of the book including DIBY, JEE Main, JEE Advanced and CTQ's.

An equation means nothing to me unless it expresses a thought of God.

–Srinivasa Ramanujan

1 SETS AND RELATIONS

HISTORICAL NOTE

Interesting Number Paradox

A Humorous paradox that arises from the attempt to classify every natural number as either 'Interesting or uniteresing"

It came into existence when G. H. Hardy told Ramanujan "The taxi cab number 1729 seems uninteresting" to which the former replied that it is the smallest number which is the sum of two cubes in two different ways.

INTRODUCTION

'A set is the mathematical model for a collection of different things; a set contains elements or members, which can be mathematical objects of any kind: number, symbols, points in space, lines, other geometrical shapes, variables or even other sets'.

Relation on a set may, or may not hold between two given members of the set. Set members may not be in relation "to a certain degree"

Relation is the branch of mathematical logic that studies sets.

☞ Set is a language of mathematics.

SET THEORY

DEFINITION

A set is a well defined collection of objects. By well defined we mean there should be no ambiguity regarding the inclusion and exclusion of the objects.

Cardinal Number

The number of distinct elements in a set A is denoted by $n(A)$ or $|A|$ and it is known as cardinal number of the set A.

REPRESENTATION OF SET

There are two methods for representing a set.

(*i*) **Tabulation method or Roster form**

All the elements belonging to the set are written in curly brackets and separated by commas

If A is the set of days of a week, then

A = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

(*ii*) **Set Builder Method or Set rule method**

In this method, we use the definition, which is satisfied by all the elements of set.

In above example set A may be written as

$A = \{x : x \text{ is a day of week}\}$

Sets

Type of Sets

Empty set or Null set

A set which has no element is called null set. It is denoted by symbol ϕ or $\{\}$.

Equivalent set

Two finite sets A and B are said to be equivalent, if $n(A) = n(B)$.

Singleton set

A set having one element is called singleton set.

Finite and Infinite set

A set which has finite number of elements is called a finite set. Otherwise, it is called an inifinite set.

Power Set

The set of all subset of a given set A is called power set of A and denoted by $P(A)$, $n(P(A)) = 2^{n(A)}$.

Equal Set

Two sets A and B are said to be equal, written as $A = B$. if every element of A is in B and every element of B is in A. Clearly equal sets are equivalent but equivalent sets need not to be equal.

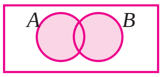
Operations of Sets

Difference of two sets

$A - B = \{x : x \in A \text{ and } x \notin B\}$ Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$

Union

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

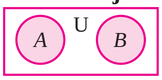


Clearly, $x \in (A \cup B) \Rightarrow x \in A$ or $x \in B$ and $x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B$

Symmetric Difference

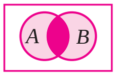
The symmetric difference of two sets A and B denoted by $A \Delta B$, is defined as $(A \Delta B) = (A - B) \cup (B - A)$

Disjoint sets



$A \cap B = \phi$,

Intersection of sets



$A \cap B$ (read as "A intersection B") and defined as $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Note

- Every set is a subset and superset of itself.
- If A is not a subset of B, we write $A \not\subset B$.
- The empty set is the subset of every set.
- If A is a set with $n(A) = n$, then number of subset of A are 2^n and the number of proper subsets of A are $2^n - 1$

Subset And Superset

If A and B are two sets such that every element of A is also an element of B, then A is a subset of B and B is superset of A. We write $A \subseteq B$.

Universal Set

The universal set is the superset for all the sets under the consideration.

The set of complex numbers is the universal set for all possible sets related to numbers.

Complement of a Sets (A' or A^c)

The comlement of a set A of all those elements of the universal set U which are not elements of A. It is denoted by A' or A^c. Clearly, A' or $A^c = U - A$.

De-Morgan's Law

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

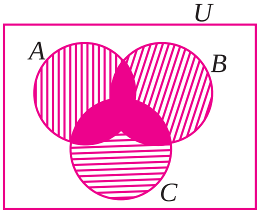
P.I.E. (Principle of Inclusion and Exclusion)

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
- $n(A_1 \cup A_2 \cup A_3 \cup A_4 \ldots A_n) = n(S_1) - n(S_2) + n(S_3) - n(S_4) \ldots (-1)^{n-1} n(S_n)$
  where $n(S_1) = n(A_1) + n(A_2) + \ldots + n(A_n)$, $n(S_2) = n(A_1 \cap A_2) + n(A_2 \cap A_3) + n(A_3 \cap A_4) + \ldots$
  $n(S_n) = n(A_1 \cap A_2 \cap A_3 \ldots \cap A_n)$

Important Results:

- $n(A - B) = n(A) - n(A \cap B)$
- $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$
- $n(A' \cup B') = n(U) - n(A \cap B)$
- $n(A' \cap B') = n(U) - n(A \cup B)$

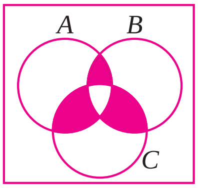
- Number of elements in exactly two of the sets A, B, C $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3\,n(A \cap B \cap C)$
- Number of elements in exactly one of the sets $A, B, C = n(A) + n(B) + n(C) - 2\,n(A \cap B) - 2\,n(B \cap C) - 2\,n(A \cap C) + 3\,n(A \cap B \cap C)$



SOLVED EXAMPLES

1. An organization awarded 48 medals in event 'A', 25 in event 'B' and 18 in event 'C'. If these medals went to total 60 men and only five men got medals in all the three events, then, how many received medals in exactly two of three events?

Sol. $n(A) = 48$, $n(B) = 25$, $n(C) = 18$



$n(A \cup B \cup C) = 60$,

$n(A \cap B \cap C) = 5$

$n(A \cup B \cup C) = \sum n(A)$

$- \sum n(A \cap B) + n(A \cap B \cap C)$

$\sum n(A \cap B) = 48 + 25 + 18 + 5 - 60 = 36$

Number of men who received exactly 2 medals

$\sum n(A \cap B) - 3n(A \cap B \cap C) = 36 - 15 = 21$

2. The number of elements in the set $\{x \in N : 10 \le x \le 100\}$ and $3^x - 3$ is a multiple of 7} is ________.

Sol. $3^x - 3$ is multiple of $7 \Rightarrow 3^x = 7\lambda + 3$

$x = 6k + 1$ form

$x = 1, 7, 13, 19........97$

Number of possible values of $x = 15$

3. Find the number of elements in the set

$\{x \in \mathbb{Z} : |x^2 - 10x + 19| < 6\}$.

Sol. $-6 < x^2 - 10x + 19 < 6$

$\Rightarrow x^2 - 10x + 25 > 0$ and $x^2 - 10x + 13 < 0$

$\Rightarrow (x - 5)^2 > 0$ and $x \in \left[5 - 2\sqrt{3}, 5 + 2\sqrt{3}\right]$

$x \in R - \{5\}$...(*i*) and $x = \{2, 3, 4, 5, 6, 7, 8\}$ ...(*ii*)

Taking intersection of (*i*) & (*ii*), we get

$\Rightarrow x = 2, 3, 4, 6, 7, 8$

$\therefore$ Number of values of $x = 6$

4. Let $A = \left\{x \in [-6, 3] - \{-2, 2\} : \dfrac{|x + 3| - 1}{|x| - 2} \ge 0\right\}$ and $B = \{x \in Z : x^2 - 7|x| + 9 \le 0\}$. Then find the number of elements of $A \cap B$.

Sol. **Case-I:** $|x| - 2 > 0$ and $|x + 3| - 1 \ge 0$

$\Rightarrow |x| > 2 \Rightarrow x > 2$ or $x < -2$

and $|x + 3| \ge 1 \Rightarrow (x + 3) \ge 1$ or $x + 3 \le -1$

$\Rightarrow x \ge -2$ or $x \le -4 \Rightarrow x \in [-6, -4] \cup (2, 3]$

Case-II: $|x| - 2 < 0$ and $|x + 3| - 1 \le 0$

$\Rightarrow |x| < 2 \Rightarrow -2 < x < 2$ and $|x + 3| \le 1$

$\Rightarrow -1 \le x + 3 \le 1 \Rightarrow -4 \le x \le -2$

$\therefore$ No common solution exists

$\Rightarrow A = \{x \in [-6, -4]\;\; (2, 3]\}$

$B = \{x \in z : x^2 - 7|x| + 9 \le 0\}$

$x^2 - 7x + 9 \le 0$ for $x \ge 0 \Rightarrow x \in \{2, 3, 4, 5\}$

$x^2 + 7x + 9 \le 0$ for $x \le 0 \Rightarrow x \in \{-5, -4, -3, -2\}$

$A = \{\pm 2, \pm 3, \pm 4, \pm 5\}$

Thus $A \cap B = \{-5, -4, 3\}$

Hence, number of elements in $A \cap B$ is 3.

5. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{3, 6, 7, 9\}$. Then find the number of elements in the set $\{C \subseteq A : C \cap B \ne \phi\}$.

Sol. $C \subseteq A$ and $C \cap B = \phi$

If C is formed only by {1, 2, 3, 4, 5, 6, 7} total number of subsets of A = 2^7.

when C is formed by{1, 2, 4, 5} = 2^4

$\therefore$ Number of subsets where $C \cap B \ne \phi$

$= 2^7 - 2^4 = 112$

6. Find the sum of all the elements of the set $\{a \in \{1, 2, ..., 100\}$: HCF $(a, 24) = 1\}$.

Sol. $\{a \in (1, 2, 3, \ldots, 100) : \text{HCF}(a, 24) = 1\}$

HCF of $(a, 24) = 1$ $\quad \therefore a = 1, 5, 7, 11, 13, 17, 19, 23$

sum of these numbers = 96.

$\therefore$ There are four such blocks and a number 97 is there upto 100.

$\therefore$ complete sum $= 96 + (24 \times 8 + 96) + (48 \times 8 + 96) + (72 \times 8 + 96) + 97 = 1633$

7. Let $A = \{n \in N$: H.C.F. $(n, 45) = 1\}$ and Let $B = \{2k : k \in \{1, 2, ..., 100\}\}$. Then find the sum of all the elements of $A \cap B$.

DIBY 1.1

1. Set A has m elements and Set B has n elements. If the total number of subsets of A is 112 more than total number of subsets of B, then find the value of $m \cdot n$.
2. Let $X = \{n \in N : 1 \le n \le 50\}$. If $A = \{n \in X: n$ is a multiple of 2); $B = \{n \in X: n$ is a multiple of 7$\}$, then find the number of elements in the smallest subset of X containing both A and B.
3. Two newspapers A and B are published in a city. It is known that 25% of the city populations reads A and 20% reads B while 8% reads both A and B. Further, 30% of those who read A but not B look into advertisements and 40% of those who read B but not A also look into advertisements, while 50% of those who read both A and B look into advertisements. Then find the percentage of the population who look into advertisement.
4. Let Z be the set of integers. If $A = \{X \in Z : 2^{(x+2)(x^2-5x+6)} = 1\}$ and $B = \{x \in Z: -3 < 2x - 1 < 9\}$, then find the number of subsets of the set $A \times B$.
5. Let $S = \{1,2,3,......, 100\}$. Find the number of non-empty subsets A of S such that the product of elements in A is even.
6. In a class of 80 students numbered 1 to 80 , all odd numbered students opt for Cricket, student whose numbers are divisible by 5 opt for Football and those whose numbers are divisible by 7 opt for Hockey. Find the number of students who do not opt any of the three games.
7. In a class of 30 pupils, 12 take needle work, 16 take physics and 18 take history. If all the 30 students take at least one subject and no one takes all three then find the number of pupils taking 2 subjects.

CARTESIAN PRODUCT OF SETS

Let A and B be any two sets. The set of all order pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B is denoted by $A \times B$. Thus, $A \times B = \{(a, b); a \in A \text{ and } b \in B\}$

If $A = \phi$ or $B = \phi$, then we defind $A \times B = \phi$

Ordered Pair

An ordered pair consists of two objects or elements in a given fixed order.

Equality of Ordered Pair

Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal iff $a_1 = a_2$ and $b_1 = b_2$, i.e. $(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$.

Properties of Cartesian Product

$A \times B = \{a, b\}$: a $\in A$ and $b \in B\}$. If A, B and C are three sets then,

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $A \times (B - C) = (A \times B) - (A \times C)$

(iv) If $A \subseteq B$ then $(A \times B) \subseteq (B \times C)$

(v) If $A \subseteq B$ then $(A \times B) \subseteq (B \times A) = A^2$

(vi) If $A \subseteq B$ and $C \subseteq D$ then $(A \times C) \subseteq (B \times D)$

(vii) $(A \times B) \cap (S \cap T) = (A \times S) \cap (B \cap T)$, where S and T are two sets.

RELATIONS

A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product $A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called the image of the first element.

Relation is denoted by aRb or $R(a, b)$.

DOMAIN OF A RELATION

The set of all first elements of the ordered pairs in a relation R form a set A to a set B is called the domain of the relation R.

RANGE OF A RELATION

The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the co-domain of the relation R.

Note: Range $\subseteq$ Co-domain.

Note

(*i*) The identity and the Universal relations on a non-void set are symmetric relations.

(*ii*) A relation R on a set A is not a symmetric relation if there are at least two elements $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

(*iii*) A reflexive relation on a set A is not necessarily symmetric. For example, the relation $R = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ is a reflexive relation on set $A = \{1, 2, 3\}$ but it is not symmetric.

Note: R = $\{(x, y): x$ is husband of $y\}$ is transitive relation.

- The void relation on a set A is symmetric and transitive
- Singleton relation is always transitive relation.
- Identity relation are always reflexive, symmetric and transitive it means they are equivalence relations.

Trick: Always look for exceptions whenever we need to check for types of relations.

Note: In general $RoS \neq SoR$, $(SoR)^{-1} = R^{-1}oS^{-1}$.

6. Transitive Relation: Let A be any set. A relation R on A is said to be a transitive relation iff $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$ i.e. aRb and $bRc \Rightarrow aRc$ for $a, b, c \in A$.

TRANSITIVE RELATION

Case-I: aRb & bRc $\Rightarrow aRc$ is transitive relation *i.e.* $R_1 = \{(1, 2), (2, 3), (1, 3)\}$

Case-II: aRb only is transitive *i.e.* $R_2 = \{(2, 3)\}$ is transitive $R_3 = \{(1, 2), (3, 2)\}$ is transitive

Case-III: aRa only is transitive *i.e.* $R_4 = \{(1, 1)\}$ is transitive $R_5 = \{(1, 1), (2, 2)\}$ is transitive

Note: The identity and the universal relations on a non-void set are transitive.

7. Equivalence Relation: A relation R on a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

8. Anti Symmetric Relation: Let A be any set. A relation R on set A is said to be an anti symmetric relation iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$.

9. Inverse Relation: If R is a relation from non empty set A to non empty set B, then the inverse relation of R is defined from the set B to A, by interchanging the first and second coordinates of ordered pairs of relations R.

If $R: A \to B$ given by, $R = \{(a, b) : a \in A$ and $b \in B\}$

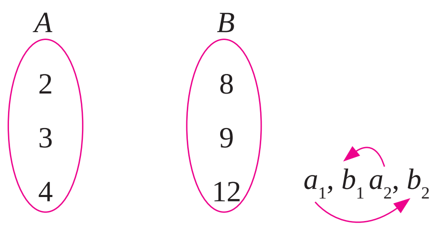
then, $R^{-1}: B \to A$ given by, $R^{-1} = \{(b, a) : b \in B$ and $a \in A\}$

10. Composition of Relations: Let R and S be two relations from sets A to B and B to C respectively. Then we can define a relation SoR from A to C such that $(a, c) \in SoR$ $\forall$ $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. The relation is called the composition or R and S.

SOLVED EXAMPLES

14. Let $A = \{2, 3, 4\}$ and $B = \{8, 9, 12\}$. Then find the number of elements in the relation $R = \{((a_1, b_1), (a_2, b_2)) \in (A \times B, A \times B) :$ a_1 divides b_2 and a_2 divides $b_1\}$.

Sol. (*a*)



a_1 divides b_2

Each element has 2 choices $\Rightarrow 3 \times 2 = 6$

a_2 divides b_1

Each element has 2 choices $\Rightarrow 3 \times 2 = 6$

Total = $6 \times 6 = 36$

15. Check the following relations for being reflexive, symmetric, transitive and thus choose the equivalence relations if any.

(*i*) $a\,R\,b$ iff $|a| \leq b$; $a, b \in$ set of real numbers.

(*ii*) $a\,R\,b$ iff $a < b$; $a, b \in$ N.

(*iii*) $a\,R\,b$ iff $|a - b| > \frac{1}{2}$; $a, b \in R$.

(*iv*) $a\,R\,b$ iff a divides b; $a, b \in N$.

DIBY 1.2

8. The relation R defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(x, y) : |x^2 - y^2| < 16\}$, then check if $R = \{(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)\}$ is a relation or not.

9. If $A = \{1, 2, 3\}$ and $R_1 = \{(1, 2), (3, 2), (1, 3)\}$, $R_2 = \{(1, 3), (3, 6), (2, 1), (1, 2)\}$, then show that R_1 is a relation and R_2 is not a relation on A.

10. Define a relation R on set $A = \{2, 3, 5, 6, 10\}$ as xRy if '$x < y$ and x divides y', then find domain of relation R.

11. A relation R is defined from $\{2, 3, 4, 5\}$ to $\{3, 6, 7, 10\}$ by $xRy \Leftrightarrow x$ is relatively prime to y. Then find domain of R.

12. Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 3, 5, 7, 9\}$. find number of relations from X to Y?

13. Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$. A relation $R : A \to B$ is defined by $R = \{(1, 3), (1, 5), (2, 1)\}$. Then find R^{-1}.

14. The relation R is defined on the set of natural numbers as $\{(a, b) : a = 2b\}$. Then find R^{-1}.

15. If R be a relation < from $A = \{1,2, 3, 4\}$ to $B = \{1, 3, 5\}$ i.e., $(a,b) \in R \Leftrightarrow a < b$, then find ROR^{-1}.

16. Let $A = \{1, 3, 6, 9\}$. Let R be the relation on A defined by $R = \{(x, y) : x \in A, y \in A$ and x divides $y\}$. Find R in roster form.

17. Find the range of the relation R given by $R = \{(x, y) : x \in N, y \in N$ and $x^2 + y^2 \le 24\}$

18. Let $A = \{2, 3, 4, 5\}$ then how many relations can be defined on set A?

19. R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by $y = x - 3$. Then find relation R^{-1}.

20. For real number x and y, we write $xRy \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then prove that the relation R is reflexive.

21. For any two real numbers a and b, we define aRb if and only if $\sin^2 a + \cos^2 b = 1$. Then prove that relation R is an equivalence relation.

22. Let S be the set of all real numbers. Then prove that the relation $R = \{(a, b): 1 + ab > 0\}$ on S is reflexive and symmetric but not transitive:

23. Let R be a relation defined on Q as follows: $a, b \in Q$, aRb if and only if $|a - b| \le 1$. Then prove that R is reflexive and symmetric?

TYPES OF RELATIONS

1. Empty Relation
A relation in which no element of A is related to any other element of A, i.e. $R = \phi \subset A \times A$.

2. Universal Relation
A relation in which each element of A is related to every element of A i.e. $R = A \times A$.

3. Identity Relation
A relation in which each element is related to itself only. $I = \{(a, a), a \in A\}$

4. Reflexive Relation
$(a, a) \in R$, for every $a \in A$
for every $a \in A$.

5. Symmetric Relation
$(a, a) \in R$ implies that
$(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

6. Transitive Relation
$(a_1, a_2) \in R$ & $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in$ R for all $a_1, a_2, a_3 \in A$

7. Equivalence Relation
A relation R in a set A is said to be on equivalence relation if R is reflexive, symmetric & transitive.

8. Inverse Relation
Inverse relation of R from A to B denoted by R^{-1}, is a relation from B to A is defined by $R^{-1} = \{(b, a)\}$

9. Anti-symmetric Relation
$(a_1, a_2) \in R$ & $(a_2, a_1) \in R$
$\Rightarrow a_1 = a_2$ where $a_1, a_2 \in A$

COUNTING OF RELATIONS

- Number of relations from set A to $B = 2^{mm}$, where $|A| = m$, $|B| = n$
- Number of symmetric relation set on a set with 'n' element $= 2^{n(n+1)/2}$
- Number of identity relation on a set with 'n' elements = 1
- Number of reflexive and symmetric relations on a set with n elements $= 2^{\frac{n(n-1)}{2}}$
- Number reflexive relation set on a set with 'n' elements $= 2^{n(n-1)}$
- Number of relations which are symmetric but not reflexive $= 2^{\frac{n(n+1)}{2}} - 2^{\frac{n(n-1)}{2}}$

REMEMBER AS A RESULT

Recall

For Transitive Relations:

If $n(A) = 1 \Rightarrow$ Number of Transitive relations = 2

If $n(A) = 2 \Rightarrow$ Number of Transitive relations = 13

If $n(A) = 3 \Rightarrow$ Number of Transitive relations = 171

If $n(A) = 4 \Rightarrow$ Number of Transitive relations = 3994

Note: There is no generic formula for above results.

For Equivalence Relations:

If $n(A) = 1 \Rightarrow$ Number of equivalence relations = 1

If $n(A) = 2 \Rightarrow$ Number of equivalence relations = 2

If $n(A) = 3 \Rightarrow$ Number of equivalence relations = 5

If $n(A) = 4 \Rightarrow$ Number of equivalence relations = 15

1. $\sum_{n=1}^{49} \frac{1}{\sqrt{n+\sqrt{n^2-1}}} = x + y\sqrt{2}$, for some integers x and y. If x denotes number of elements in set A & y denote number of elements in set B. Find the number of relations defined from A to B.

Sol. We have $\frac{1}{\sqrt{n+\sqrt{n^2-1}}} = \frac{1}{\sqrt{\left(\sqrt{\frac{n+1}{2}}-\sqrt{\frac{n-1}{2}}\right)^2}}$

$$= \frac{1}{\sqrt{\frac{n+1}{2}}+\sqrt{\frac{n-1}{2}}} = \frac{\sqrt{\frac{n+1}{2}}-\sqrt{\frac{n-1}{2}}}{\frac{n+1}{2}-\frac{n-1}{2}}$$

$$= \sqrt{\frac{n+1}{2}}-\sqrt{\frac{n-1}{2}}.$$

Hence the sum from the statement

$$\sqrt{\frac{49+1}{2}}+\sqrt{\frac{48+1}{2}}-\sqrt{\frac{1}{2}}-0 = 5+\frac{7}{\sqrt{2}}-\frac{1}{\sqrt{2}} = 5+3\sqrt{2}.$$

Number of relations = $2^{3\times5} = 2^{15}$.

2. Let $S = \{(a, b, c)\} \in N \times N \times N : a + b + c = 21, a \le b \le c\}$ and $T = \{(a, b, c)\} \in N \times N \times N : a, b, c$ are in A.P$\}$, where N is the set of all natural numbers. Then find the number of elements in the set $S \cap T$.

Sol. $a + b + c = 21$ and $b = \frac{a+c}{2} \Rightarrow a + c = 14$ and $b = 7$

So, a can take values from 1 to 6, when c range from 13 to 8, or $a = b = c = 7$

So, 7 triplets.

3. The number of 3 digit numbers, that are divisible by singleton element of set A or element of set B but not divisible by elements of set C. If $aN = \{ax : x \in N\}$ and $A = 3N$, $B = 4N$, $C = 48N$.

Sol. Total number of 3 digit number = 999 – 99 = 900

Number of 3-digit number which are divisible by 3 $= 300\left(\text{Using } \frac{900}{3} = 300\right)$

Number of 3-digit number which are divisible by 4 $= 225\left(\text{Using } \frac{900}{4} = 225\right)$

Number of 3-digit number which are divisible by 3 and 4 both = 75

$\left(\text{Using } \frac{900}{12} = 75\right)$

Number of 3-digit number which are divisible by either 3 or 4

= 300 + 225 – 75 = 450

We have to remove divisible by 48,

144, 192, ..., 18 terms

Required number of 3-digit number which are divisible by 3 or 4 but not 48 = 450 – 18 = 432.

4. Let $A = \{-4, -3, -2, 0, 1, 3, 4\}$ and $R = \{(a, b) \in A \times A : b = |a|$ or $b^2 = a + 1\}$ be a relation on A. Then find the minimum number of elements, that must be added to the relation R so that it becomes reflexive and symmetric.

Sol. R = [(–4, 4), (3, 3), (3, –2), (0, 1), (0, 0), (1, 1), (4, 4), (3, 3)}

For reflexive, add $\Rightarrow$ (–2, –2), (4, 4), (–3, –3)

For symmetric, add $\Rightarrow$ (4, 4), (3, 3), (–2, 3), (1, 0).

Minimum number of elements = 7.

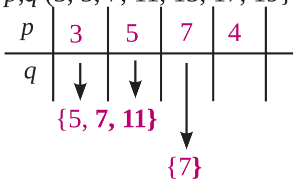
5. Let R be a relation from the set {1,2,3. ...., 60} to itself such that $R = \{(a,b):b = pq$ where $p,q \ge 3$ are prime numbers}. Then find the number of elements in R.

Sol. $a, b \in \{1,2,3, ..., 60\}$

$p,q \in$ {3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59) ⟶ total 16

$$b = pq \le 60 \begin{cases} p \le 20 \Rightarrow p \le 19 \\ q \le 20 \Rightarrow q \le 19 \end{cases} \text{ as } pq \le 60 \;\&\begin{cases} p \ge 3 \\ q \ge 3 \end{cases}$$

p,q (3, 5, 7, 11, 13, 17, 19}

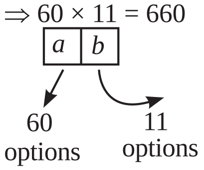


(3, 5, 7, 11, 13, 17, 19}

7 + 3 + 1 = 11

$R = \{(a,b) : b = pq$, where $p,q \ge 3$ are prime}

$\Rightarrow 60 \times 11 = 660$



6. Let a set $A = A_1 \cup A_2 \cup .... \cup A_k$, where $A_i \cap A_j = \phi$ for $i \ne j$, $1 \le i, j \le k$. Define the relation R from A to A by $R = \{(x, y) : y \in A_i$ if and only if $x \in A_i$, $1 \le i \le k\}$. Then show that R is an equivalence relation.

Sol. Since, aRb = a is related to b, belongs to A iff a belongs to A.

For reflexive: aRa, $a \in A$, so it is true.

For symmetric : a & b belongs to the same set.

$\Rightarrow b$ & a also belongs to the same set$\Rightarrow$ bRa will be true

For transitive: $aRb \Rightarrow b, c$ belongs to the same set

$bRc \Rightarrow b, c$ belongs to the same set

$\Rightarrow (a, c)$ belongs to the same set $\Rightarrow$ so aRc will be true.

So R is an equivalence relation.

SINGLE CORRECT

1. Let R be a relation in N defined by

$R = \{(1 + x, 1+x^2): x \leq 5, x \in N\}$. Which of the following is false?

(*a*) R = {(2, 2), (3, 5), (4, 10), (5, 17), (6, 25)}

(*b*) Domain of R = {2, 3, 4, 5, 6}

(*c*) Range of R = {2, 5, 10, 17, 26}

(*d*) None of these

2. The domain and range of the relation R given by

$R = \{(x, y): y = x + \frac{6}{x}$ where $x, y \in N$ and $x < 6\}$, respectively, is:

(*a*) {1, 2, 3}, {7, 5} (*b*) {1, 2}, {7, 5}

(*c*) {2, 3}, {5} (*d*) {2, 3}, {5, 3}

3. Consider the following with regard to a relation R on a set of real numbers defined by xRy if and only if $3x + 4y = 5$. Consider the following three statements:

(1) $0\,R1$ (2) $1R\frac{1}{2}$ (3) $\frac{2}{3}R\frac{3}{4}$

Which of the above are correct?

(*a*) 1 and 2 only (*b*) 1 and 3 only

(*c*) 2 and 3 only (*d*) 1, 2 and 3

4. If $R = \{(x, y) \mid x, y \in Z, x^2 + y^2 \leq 4\}$ is a relation on Z, then domain of R is:

(*a*) {0, 1, 2} (*b*) {0, –1, –2}

(*c*) {–2, –1, 0, 1, 2} (*d*) None of these

5. The linear relation between the components of the ordered pairs of relation R given by:

$R = \{(0, 2), (–1, 5), (2, –4), \ldots\ldots\ldots\}$ is

(*a*) $x + y = 2$ (*b*) $3x – y = 1$

(*c*) $x + 3y = 2$ (*d*) $3x + y = 2$

6. Let A be the set of first ten natural numbers and let R be a relation on A defined by $(x, y) \in R \Leftrightarrow x + 2y = 10$, i.e., $R = \{(x, y) : x \in A, y \in A$ and $x + 2y = 10\}$. Then the domain and range of R is

(*a*) {2, 4, 6, 8}, {4, 3, 2, 1} respectively

(*b*) {4, 3, 2, 1}, {2, 4, 6, 8} respectively

(*c*) {1, 2, 3, 4}, {1, 2, 3, 4} respectively

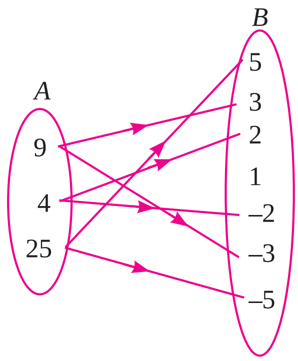
(*d*) None of these

7. Define relation R_1 and R_2 on set A = {2, 3, 5, 7, 10} as xR_1y if $x \mid (y – 1)$ and xR_2y of $x + y = 10$) then the relation R given by $R = R_1 \cap R_2$ is

(*a*) {(2, 4)} (*b*) {(3, 7)}

(*c*) {(3, 7), (5, 5)} (*d*) None of these

8. The figure given below shows a relation R between the sets A and B.



Then which of the following is correct?

I. The relation R in set builder form is $\{(x, y) : x$ is the square of $y, x \in A, y \in B\}$.

II. The domain of the relation R is {4, 9, 25}

III. The range of the relation R is {–5, –3, –2, 2, 3, 5}

(*a*) Only I and II are true

(*b*) Only II and III are true

(*c*) I, II and III are true

(*d*) Neither I, II nor III are true

9. The relation R_1 and R_2 are defined from R to R as given below (R stand for set of real numbers)

$R_1 = \{(x, y) : |x – 3| \leq 1, |y – 3| \leq 1\}$ and

$R_2 = \{(x, y) : 4x^2 + 9y^2 – 32x – 54y + 109 \leq 0\}$

Choose the correct option

(*a*) $R_1 \subset R_2$ (*b*) $R_1 = R_2$

(*c*) $R_2 \subset R_1$ (*d*) None of these

10. On Q, the set of rational numbers, define a relation R as follows:

aRb is $a \cos 15^\circ + b \sin 15^\circ$ is an irrational number, then:

(*a*) domain of R is Q

(*b*) domain of R is $Q – Z$

(*c*) domain of R is $Q – N$

(*d*) domain of R is $Q – A$ where A is a singleton set.

11. Let a relation R be defined by R = {(4, 5), (1, 4), (4, 6), (7, 6), (3, 7)} then R^{-1}oR is:

(*a*) {(1, 1), (4, 4), (4, 7), (7, 4), (7, 7), (3, 3)}

(*b*) {(1, 1), (4, 4), (7, 7), (3, 3)}

(*c*) {(1, 5), (1, 6), (3, 6)}

(*d*) None of these

12. If R is a relation from a set A to set B, then:

(*a*) $R \subset B \times A$ (*b*) $R \subset A \times B$

(*c*) $R = A \times B$ (*d*) $A \times B \subset R$

CRITICAL THINKING QUESTIONS (CTQ'S)

1. Determine all integer $n \geq 1$ for which there exists a pair of positive integers (a, b) such that no cube of a prime divides $a^2 + b + 3$ and

$$A = \left\{ n : \frac{ab + 3b + 8}{a^2 + b + 3} = n ; n \in N,\ a, b \in N \right\}.$$

2. Find the cardinal number of set of positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, ... d_k)$ such that for every $i = 1, 2, ... k$, the number $d_1 + ... + d_i$ is a perfect square.

3. Find sum of all pairs (p, q), with $p > q$ for which the number

$$S = \left\{ (p, q); \frac{(p+q)^{p+q}(p-q)^{p-q} - 1}{(p+q)^{p-q}(p-q)^{p+q} - 1} = n, n \in Z \right\}.$$

4. A relation $(a,b)Rc,d)$ is defined on 4-tuples (a, b, c, d) of natural numbers with $a < b < c$ and

$a! + b! + c! = 3^d$. Find the number of tuples.

5. A set $S = \{(p, q, r): pq = r + 1, 2(p^2 + q^2) = r^2 + 1$, where p, q, r are primes$\}$.

Find the cardinal number of S.

6. There is a set $S = \{(a, b, c): a^2 = bc + 1, b^2 = ca + 1, a, b, c \in Z)\}$. Find the cardinal number of S.

7. For any natural number n, expressed in base 10, let $S(n)$ denote the sum of all digits of n. Find all natural numbers n such that

$n^3 = 8S(n)^3 + 6nS(n) + 1$.

8. There is a set $S = \{(m, n): m^5 - n^5 = 16mn, m, n \in Z\}$. A relation is define on (m, n) such that

$R(m, n: m + n = 0)$.

Then find sum of number of transitive and symmetric relations in R.

9. A set S is defined such that

$S = \{(p, q): 1 + \dfrac{p^q - q^p}{p+q}$ is a prime number, p, q are prime numbers$\}$.

A relation R is defined such that $R = (p, q)$. Find number of transitive relations in R.

10. Let $A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ and $B = A - I$. If $\omega = \dfrac{\sqrt{3}i - 1}{2}$, then the number of elements in the set

$\{n \in \{1,2,...,100\} : A^n + (\omega B)^n = A + B\}$ is equal to......

11. Let S denote the set of all 6-tuples (a, b, c, d, e, f) of positive integers such that $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$. Consider the set $T = \{abcdef : (a, b, c, d, e, f) \in S\}$. Find the greatest common divisor of all the members of T.

ANSWER KEY

DIBY-1.1

1. [28] **2.** [29] **3.** [13.9] **4.** [2^{15}] **5.** [$2^{100} - 2^{50}$] **6.** [28] **7.** [16]

DIBY-1.2

8. [Relation] **10.** $R = \{2, 3, 5\}$ **11.** $R = \{2, 3, 4, 5\}$

12. [2^{25}] **13.** $R^{-1} = \{(3, 1), (5, 1), (1, 2)\}$ **14.** [$R^{-1} = \{(1, 2), (2, 4), (3, 6) ......\}$]

15. [$RoR^{-1} = \{(3, 3), (3, 5), (5, 3), (5, 5)\}$] **16.** [$R = \{(1, 1), (1, 3), (1, 6), (1, 9), (3, 3), (3, 6), (3, 9), (6, 6), (9, 9)\}$]

17. {1, 2, 3, 4} **18.** [$(16)^4$] **19.** {(8, 11), (10, 13)}

JEE MAIN

1. (*a*) **2.** (*a*) **3.** (*c*) **4.** (*c*) **5.** (*d*) **6.** (*a*) **7.** (*b*) **8.** (*c*) **9.** (*a*) **10.** (*a*)

11. (*a*) **12.** (*b*) **13.** (*a*) **14.** (*b*) **15.** (*d*) **16.** (*d*) **17.** (*c*) **18.** (*c*) **19.** (*c*) **20.** (*a*)

21. (*d*) **22.** (*b*) **23.** (*a*) **24.** (*d*) **25.** (*b*) **26.** (*a*) **27.** (*a*) **28.** (*b*) **29.** (*d*) **30.** (*b*)

31. (*b*) **32.** (*b*) **33.** [27] **34.** (*a*) **35.** (*a*) **36.** (*d*) **37.** (*a*) **38.** (*d*) **39.** [13] **40.** [6]

41. [18] **42.** [19] **43.** [3] **44.** [7] **45.** [832] **46.** [1251] **47.** [30] **48.** [38]

CRITICAL THINKING QUESTIONS (CTQ'S)

1. [2] **2.** [2] **3.** [5] **4.** [3] **5.** [2] **6.** [8] **7.** [17] **8.** [3] **9.** [1] **10.** [17]

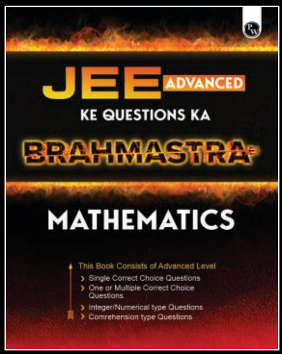
11. [24]

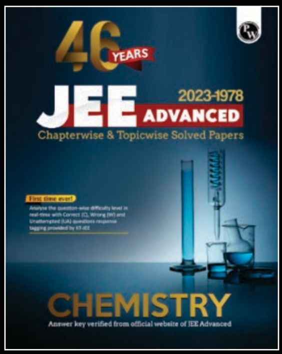
About Author

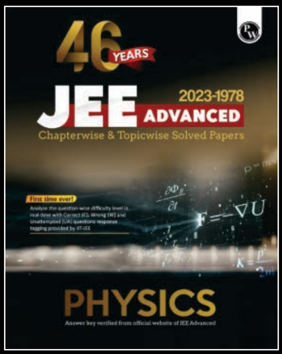
Sachin Jakhar, graduated in Electrical Engineering from NIT Kurukshetra. He has been nurturing students through offline & online platforms for over a decade. Due to his keen interest in Mathematics, he developed an interesting, engaging & unique style of teaching, which helps thousands of student to get in IIT's , NIT's or elite Engineering Institutes of India. Sachin Sir has the ability to inspire & motivate students to surpass their limits to achieve desired goals & success.

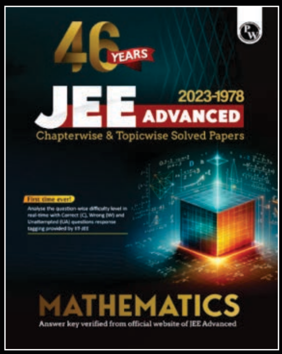
I want to dedicate this book to countless students who have embraced me with their love & appreciated my teaching approach. Love you all!!

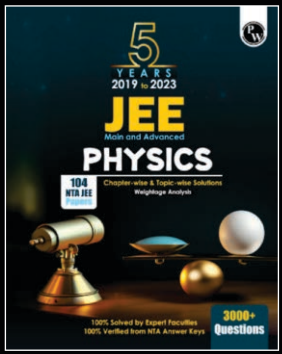
Other Important Books

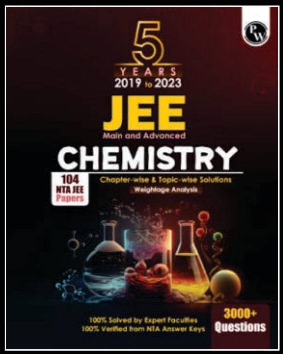


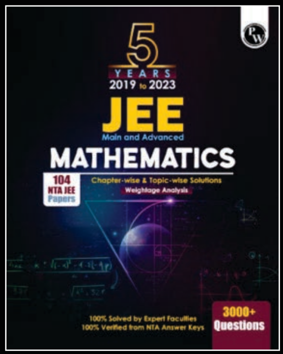












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